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Relativistic superfluids and Cosserat continua

G. Ferrarese

Dipartimento di Matematica, Università "La Sapienza", Piazzale A. Moro, 5, 00185 Roma, Italy

Abstract

We use a *generalized frame of reference* in the polar sense and the connected non-holonomic techniques to study *superfluids in general relativity* and obtain a *precise analogy between superfluids and Cosserat continua*, from both the descriptive and the structural points of view.

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À A. Lichnerowicz à l'occasion de son 80-ème anniversaire

1. Introduction

In general relativity the standard notion of frame of reference is generally assumed to be a *local orthogonal* 1×3 *structure* ($1 =$ time and $3 =$ space) (see [1, Chap. IV]), so that it is characterized by a unit time-like vector field γ or by the congruence of its flow lines Γ.

From a kinematical point of view, Γ defines a *standard continuum*, whereas the space–time gradient of γ summarizes its three fundamental characters: $c^2 C_i$ (*4-acceleration*), ω_{ik} (*proper vortex*) and k_{ik} (*proper deformation rate*).

The preceding characterization can be generalized, in a natural way, according to the theory of *polar continua* [2], by considering a *local non-orthogonal* 1×3 *structure*. More precisely, in this context, the space–time is supposed to be provided with two independent geometrical ingredients:

(i) a *time-oriented congruence* Γ, with a unitary tangent vector field γ:

$$\gamma \cdot \gamma = -1, \tag{1}$$

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(ii) a space-like *distribution of 3-planes*: $\hat{\Sigma}$, non-orthogonal to γ, namely a second time-like vector field $\boldsymbol{\eta}$ (tangent to the normal congruence to such 3-planes) which, without any loss of generality, can be chosen so that

$$\gamma \cdot \boldsymbol{\eta} = -1. \tag{2}$$

The $(\Gamma, \hat{\Sigma})$ *non-orthogonal quasi-product* structure can be introduced, more generally, directly in a (non-Riemannian) *differential manifold*, by means of *two vector fields – one contravariant*: γ^α and the other *covariant*: η_α, both defined up to a multiplicative scalar factor and the condition: $\gamma^\alpha \eta_\alpha \neq 0$ [3]. However, the most interesting situations belong to the *Riemannian case*.

Examples of this structure occur, for instance, in the ordinary three-dimensional space when one studies the problem of the evolution of a surface (*wave*) with the associated *rays*, generally non-orthogonal to the wavefronts, as well as in the *dynamics of a holonomic system*, with time-dependent constraints. Here in fact the events space, E_{n+1}, is naturally provided with the *non-orthogonal structure*, defined by the temporal lines $x^0 =$ var. and the n-manifolds $x^0 =$ const., respectively [4]. Of course, in both the considered cases, the $\hat{\Sigma}$-distribution is integrable, and the framework is non-relativistic.

Condition (2) does not exclude the possibility of *isotropy* for the field $\boldsymbol{\eta}$: $\boldsymbol{\eta} \cdot \boldsymbol{\eta} = 0$ (*null congruence*), a case in which, for every point $E \in V_4$, the 3-plane $\hat{\Sigma}$ is tangent along $\boldsymbol{\eta}$ to the light cone.

Therefore, the structure of generalized frame of reference can be used also to study *null congruences*: now the field $\boldsymbol{\eta}$ defines the frame of reference in the standard sense, and, in addition, we have a *light-like field* $\boldsymbol{l} = \boldsymbol{\eta}$, with the relative normal 3-plane $\hat{\Sigma}$, of *parabolic type*, containing the vector $\boldsymbol{l}$ (see [5]). However, the lines of the field $\boldsymbol{\eta}$ constitute a second time-like congruence $\hat{\Gamma}$, different from Γ; so, at least from the descriptive point of view, we have a *first analogy between generalized frames of reference and superfluids* or *binary fluid mixtures*, described by means of two different time-like congruences.

A second analogy arises by considering the generalized frames of reference and the associated non-holonomic techniques; it can be shown that:

Proposition. *From the description point of view, every superfluid is equivalent to a cosserat continua.*

From the dynamical point of view, instead, the equivalence between the two types of continuous systems, *binary mixtures* and *Cosserat continua*, involves other interesting points of view and possible alternatives; anyway, here we will restrict our approach to the geometric-kinematical aspects only.

2. Digression on the generalized frames of reference: Quasi-natural basis

Let us suppose the space–time V_4 to be provided with a fixed generalized frame of reference $(\Gamma, \hat{\Sigma})$, or in other words with two time-like vector fields $(\gamma, \boldsymbol{\eta})$ satisfying conditions

(1) and (2); as in the ordinary case ($\boldsymbol{\eta}$ absent) we furthermore assume the *coordinates* (x^α) *to be adapted* to the congruence Γ, in the sense that

$$\boldsymbol{\gamma} = \gamma^0 \mathbf{e}_0, \quad \gamma^0 > 0.$$

Such coordinates are defined up to transformations $x^\alpha \to x^{\alpha'}$ of the kind

$$x^{0'} = x^{0'}(\boldsymbol{x}), \qquad x^{i'} = x^{i'}(x^1, x^2, x^3) \quad (i = 1, 2, 3), \tag{3}$$

where the functions on the right-hand side (which must be invertible) satisfy the conditions

$$\frac{\partial x^{0'}}{\partial x^0} > 0, \qquad \det \left\| \frac{\partial x^{i'}}{\partial x^i} \right\| > 0, \tag{4}$$

which preserve the orientation in both time and space.

Transformations (3) constitute the *fundamental continuous group* associated to the frame of reference, even if such a group depends only on the Γ component.

Associated to the coordinates (x^α) we have the natural (coordinate) *basis* $\{\mathbf{e}_\alpha\}$ and its *dual* $\{\mathbf{e}^\alpha\}$, which are *non-adapted* to the generalized frame of reference; it is convenient instead to choose a *non-holonomic distribution* $\{\tilde{\mathbf{e}}_\alpha\} \sim \{\tilde{\mathbf{e}}^\alpha\}$ *adapted to* $(\Gamma, \hat{\Sigma})$. Without any loss of generality, we will assume the following *quasi-natural distribution*:

$$\tilde{\mathbf{e}}_0 = \boldsymbol{\gamma}, \qquad \tilde{\mathbf{e}}_i = \mathbf{e}_i - \frac{\eta_i}{\eta_0}\mathbf{e}_0 \in \hat{\Sigma}, \tag{5}$$

which is *canonically associated to the coordinates* (x^α); [1] for an arbitrary change of coordinates (3), i.e. *internal* to the frame of reference, the following transformation law holds:

$$\tilde{\mathbf{e}}_{0'} = \tilde{\mathbf{e}}_0 = \text{inv.}, \qquad \tilde{\mathbf{e}}_{i'} = \frac{\partial x^i}{\partial x^{i'}}\tilde{\mathbf{e}}_i, \tag{6}$$

as it is *typical for holonomic basis.*

In the following, we will uniform the notation by adopting a "$\hat{\ }$" for all quantities relative to $\hat{\Sigma}$, in particular we write

$$\hat{\mathbf{e}}_i \stackrel{\text{def}}{=} \tilde{\mathbf{e}}_i, \qquad \hat{\gamma}_i \stackrel{\text{def}}{=} \tilde{\gamma}_i \equiv \boldsymbol{\gamma} \cdot \tilde{\mathbf{e}}_i, \tag{7}$$

and denote by $\{\hat{\mathbf{e}}^i\}$ the *dual basis* of $\{\hat{\mathbf{e}}_i\}$ on $\hat{\Sigma}$, which is characterized by the *reciprocity conditions*

$$\hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}_k = \delta^i_k, \tag{8}$$

moreover we denote by $\hat{\gamma}_{ik}$ the *induced metrics* on $\hat{\Sigma}$:

$$\hat{\gamma}_{ik} \stackrel{\text{def}}{=} \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k, \tag{9}$$

and by $\hat{\gamma}^{ik}$ the *dual metrics*, which necessarily is of the kind

$$\hat{\gamma}^{ik} = \hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}^k. \tag{10}$$

[1] An analogous distribution follows from the *dual basis* $\{\tilde{\mathbf{e}}^\alpha\}$: $\tilde{\mathbf{e}}^0 = \boldsymbol{\gamma}$, $\tilde{\mathbf{e}}^i = \mathbf{e}_i - (\eta^i/\eta^0)\mathbf{e}^0$.

As for the *metrics of* V_4, in non-holonomic terms, it is characterized by the products $\tilde{g}_{\alpha\beta} = \tilde{\mathbf{e}}_\alpha \cdot \tilde{\mathbf{e}}_\beta$:

$$\tilde{g}_{00} = -1, \qquad \tilde{g}_{0i} = \hat{\gamma}_i, \qquad \tilde{g}_{ik} = \hat{\gamma}_{ik}, \tag{11}$$

where the components $\hat{\gamma}_i$, given by the second equation in (7), coincide with *the difference of the two vector fields* $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$:

$$\hat{\gamma}_i = \gamma_i - \eta_i, \qquad \gamma_0 - \eta_0 = 0. \tag{12}$$

In contravariant form, (5) is equivalent to the following relations:

$$\tilde{\mathbf{e}}^0 = -\boldsymbol{\eta}, \qquad \tilde{\mathbf{e}}^i = \mathbf{e}^i; \tag{13}$$

we then have the *dual metrics* $\tilde{g}^{\alpha\beta} = \tilde{\mathbf{e}}^\alpha \cdot \tilde{\mathbf{e}}^\beta$:

$$\tilde{g}^{00} = \boldsymbol{\eta} \cdot \boldsymbol{\eta} = -\eta^2, \qquad \tilde{g}^{0i} = -\eta^i, \qquad \tilde{g}^{ik} = \gamma^{ik}, \tag{14}$$

where the coefficients η^2, η^i and γ^{ik} are given by the formulae

$$\eta^2 = (1 + \hat{\gamma}^2)^{-1}, \qquad \eta^i = -\eta^2 \hat{\gamma}^i, \qquad \gamma^{ik} = \hat{\gamma}^{ik} - \eta^2 \hat{\gamma}^i \hat{\gamma}^k, \tag{15}$$

in which $\hat{\gamma}^2$ is the *norm of the field* $\hat{\boldsymbol{\gamma}} \in \hat{\Sigma}$:

$$\hat{\gamma}^2 = \hat{\gamma}_i \hat{\gamma}^i \equiv \hat{\gamma}_{ik} \hat{\gamma}^i \hat{\gamma}^k. \tag{16}$$

3. Commutation formulae and non-holonomy tensor. Jacobi's identities

The non-holonomic distribution (5) gives rise to the following *pfaffian derivatives* $\tilde{\partial}_\alpha$:

$$\tilde{\partial}_0 \equiv \partial = \gamma^0 \frac{\partial}{\partial x^0}, \qquad \tilde{\partial}_i = \frac{\partial}{\partial x^i} - \frac{\eta_i}{\eta_0} \frac{\partial}{\partial x^0}, \tag{17}$$

which only depend on the field η_α, according to the choice of the coordinates: $\gamma^0 = -1/\gamma_0$, $\gamma_0 = \eta_0$.

Of course such derivatives do not commute:

$$[\tilde{\partial}_\alpha, \tilde{\partial}_\beta] = \tilde{A}_{\alpha\beta}{}^\rho \tilde{\partial}_\rho \tag{18}$$

and *the non-holonomy tensor* $\tilde{A}_{\alpha\beta}{}^\rho$ *of the distribution* (5) *is non-vanishing.*

More precisely, we have the following *commutation formulae*, like in the ordinary case:

$$[\partial, \tilde{\partial}_i] = \hat{C}_i \partial, \qquad [\tilde{\partial}_i, \tilde{\partial}_k] = 2\hat{\Omega}_{ik} \partial, \tag{19}$$

they involve *two fundamental geometric ingredients of* $\hat{\Sigma}$: $\hat{C}_i$ and $\hat{\Omega}_{ik}$ (*skewsymmetric*), which have the usual expressions, except for $\boldsymbol{\gamma}$ which is replaced by $\boldsymbol{\eta}$:

$$\hat{C}_i \stackrel{\text{def}}{=} \tilde{\partial}_i \log(-\eta_0) + \eta_0 \partial \frac{\eta_i}{\eta_0}, \qquad \hat{\Omega}_{ik} \stackrel{\text{def}}{=} \frac{1}{2} \eta_0 \left(\tilde{\partial}_i \frac{\eta_k}{\eta_0} - \tilde{\partial}_k \frac{\eta_i}{\eta_0} \right). \tag{20}$$

Of course, we have not the usual geometric meaning, because $\hat{C}_i$ *coincides with the spatial part* (on $\hat{\Sigma}$) *of the Lie derivative of* $\boldsymbol{\eta}$ *along* $\boldsymbol{\gamma}$ [3, p. 75]; instead *the tensor* $\hat{\Omega}_{ik}$ *characterizes,* *by the condition* $\hat{\Omega}_{ik} = 0$ (not *normal congruences* Γ any more, but), *distributions* $\{\hat{\Sigma}\}$ *which are holonomic*, i.e. *tangent* to a ∞^1 family of spatial manifolds V_3 (i.e. $\hat{\Omega}_{ik} = 0 \Longrightarrow \hat{T} = \text{normal}$).

From the kinematical point of view, the tensor $(c/\eta)\hat{\Omega}_{ik}$ (or better its adjoint on $\hat{\Sigma}$) can still be interpreted as *the proper vortex* of the generalized continuum; however, such a vortex defines the *free angular velocity*, which is typical of the cosserat continua. In other words, $\hat{\Omega}_{ik}$ *is not determined by the field of the velocities of the continuum*: $\mathbf{V} = c\gamma$, which instead gives a different angular velocity.

Anyway, in a generalized frame of reference, the field $\hat{\gamma}_i$ plays a fundamental role, in fixing the spatial platform $\hat{\Sigma}$; of course $\hat{\gamma}_i$ *is independent of the relative gravitational potentials*: $\eta_0 < 0, \eta_i$ and γ_{ik}, which generate, by derivation, the *gravitational field* (20) and the *deformation field*: $\hat{K}_{ik} \stackrel{\text{def}}{=} \frac{1}{2}\partial\hat{\gamma}_{ik}$, respectively.

Moreover, from the relative point of view, also in a generalized frame of reference, *the effective gravitational potentials are nine* (not 10); because in the *general case*, i.e. without a priori hypothesis about the metric structure of V_4 (stationariness, sphericity, etc.) one can always suppose $\gamma_0 = -1$ [1, p. 309].

However, different from the absolute formulation, *the relative formulation of the gravitational equations reduces the unknown quantities by one unity*, like in the *relative dynamics of particles with scalar structure* m_0 *(proper mass)*, where the relative formulation depends on *seven essential variables*: x^i, p_i and m_0, *and not on eight* as in the absolute formulation: x^α and P_α.

In any case, for every non-holonomic distribution $\{\tilde{\mathbf{e}}_\alpha\}$, we have not only the commutation formulae (18), but also the *Jacobi's identities* for the brackets $[\tilde{\partial}_\alpha, \tilde{\partial}_\beta]$; so we can deduce the following differential conditions for the non-holonomy tensor $\tilde{A}_{\alpha\beta}{}^\rho$:

$$\partial_{[\rho}\tilde{A}_{\alpha\beta]}{}^{\sigma} - \tilde{A}_{\nu[\rho}{}^{\sigma}\tilde{A}_{\alpha\beta]}{}^{\nu} = 0. \tag{21}$$

In our case, according to (19), the only non-trivial components of the non-holonomy tensor are: $\tilde{A}^0_{0i} = \hat{C}_i$ and $\tilde{A}^0_{ik} = 2\hat{\Omega}_{ik}$; so (21) is equivalent to the following *conditions for the fields* $\hat{C}_i$ and $\hat{\Omega}_{ik}$:

$$\partial\hat{\Omega}_{ik} = \tilde{\partial}_{[i}\hat{C}_{k]}, \qquad \tilde{\partial}_{[i}\hat{\Omega}_{kh]} - \hat{C}_{[i}\hat{\Omega}_{kh]} = 0. \tag{22}$$

Of course (22) reduces to trivial identities if both the tensors $\hat{C}_i$ and $\hat{\Omega}_{ik}$ are expressed in function of the potentials η_α, by means of relations (20).

4. Fundamental pfaffian derivatives and Ricci rotation coefficients

Let us now consider the derivatives of the fundamental fields $\tilde{\mathbf{e}}_\alpha$, in order to extend the pfaffian operator $\tilde{\partial}_\alpha$ to every tensor field; first of all we have

$$\tilde{\partial}_\alpha\tilde{\mathbf{e}}_\beta = \tilde{\mathcal{R}}_{\alpha\beta}{}^{\rho}\tilde{\mathbf{e}}_\rho, \tag{23}$$

where $\tilde{\mathcal{R}}_{\alpha\beta}{}^{\rho}$ are the *Ricci rotation coefficients* associated to the assigned non-holonomic distribution $\{\tilde{\mathbf{e}}_\alpha\} \sim \{\tilde{\mathbf{e}}^\alpha\}$. Such coefficients can be expressed in compact form, by means of the *Christoffel symbols* and the *non-holonomy tensor*:

$$\tilde{\Gamma}_{\alpha\beta,\sigma} \stackrel{\text{def}}{=} \tfrac{1}{2}(\tilde{\partial}_\alpha \tilde{g}_{\beta\sigma} + \tilde{\partial}_\beta \tilde{g}_{\sigma\alpha} - \tilde{\partial}_\sigma \tilde{g}_{\alpha\beta}), \qquad \tilde{A}_{\alpha\beta,\sigma} \stackrel{\text{def}}{=} \tilde{g}_{\sigma\rho} \tilde{A}_{\alpha\beta}{}^{\rho}, \tag{24}$$

indeed, the following *general formula* holds [3, p. 69]:

$$\tilde{\mathcal{R}}_{\alpha\beta}{}^{\rho} = \tilde{g}^{\rho\sigma}[\tilde{\Gamma}_{\alpha\beta,\sigma} + \tfrac{1}{2}(\tilde{A}_{\sigma\alpha,\beta} + \tilde{A}_{\alpha\beta,\sigma} - \tilde{A}_{\beta\sigma,\alpha})]. \tag{25}$$

Of course, we can also follow a direct way, explicitly deducing, by a geometrical approach, the ∂ and $\tilde{\partial}_i$ derivatives of the fields γ and $\tilde{\mathbf{e}}_i$. In both the cases, we have the following *fundamental relations*:

$$\begin{aligned} \partial\gamma &= \hat{C}^i(\hat{\mathbf{e}}_i + \hat{\gamma}_i\gamma), & \tilde{\partial}_i\gamma &= \hat{H}_i{}^k(\hat{\mathbf{e}}_k + \hat{\gamma}_k\gamma), \\ \partial\hat{\mathbf{e}}_i &= \hat{H}_i{}^k\hat{\mathbf{e}}_k + \hat{K}_i\gamma, & \tilde{\partial}_i\hat{\mathbf{e}}_k &= \hat{\mathcal{R}}_{ik}{}^j\hat{\mathbf{e}}_j + \hat{\mathcal{R}}_{ik}\gamma, \end{aligned} \tag{26}$$

where the fields

$$\mathbf{C} = \hat{C}^i\hat{\mathbf{e}}_i, \qquad \mathbf{H}_i \stackrel{\text{def}}{=} \hat{H}_i{}^k\hat{\mathbf{e}}_k \tag{27}$$

summarize the *curvature of the lines of Γ* and the *proper angular and deformation velocities* of the generalized frame of reference, respectively; instead the fields $\hat{K}_i$ and $\hat{\mathcal{R}}_{ik}$ have the following form:

$$\hat{K}_i = \hat{H}_i{}^k\hat{\gamma}_k + \hat{\complement}_i, \qquad \hat{\mathcal{R}}_{ik} = H_{ik} - \tilde{\nabla}_i\hat{\gamma}_k \tag{28}$$

with

$$H_{ik} \stackrel{\text{def}}{=} \hat{H}_i{}^j\gamma_{jk}, \qquad \gamma_{jk} \stackrel{\text{def}}{=} \hat{\gamma}_{jk} + \hat{\gamma}_j\hat{\gamma}_k, \tag{29}$$

where $\tilde{\nabla}_i$ is the *covariant extension of the pfaffan derivatives* $\tilde{\partial}_i$, obtained by means of the *Ricci rotation coefficients* $\hat{\mathcal{R}}_{ik}{}^j$, which are symmetric with respect to the indices i and k.

We notice that, except for the spatial field $\hat{\gamma}_i$, *both the ingredients $\hat{K}_i$ and $\hat{\mathcal{R}}_{ik}$ are well determined functions of the characteristic fields* (27); indeed the following relation holds:

$$\hat{\complement}_i = C_i - \partial\hat{\gamma}_i, \qquad \text{where } C_i \stackrel{\text{def}}{=} \gamma_{ik}\hat{C}^k. \tag{30}$$

Anyway, in (26), the geometric-kinematic meaning of the coefficients H_{ik} and $\hat{\mathcal{R}}_{ik}{}^j$ is not yet specified.

As for the field H_{ik}, according to what is previously said, the two parts, skewsymmetric $H_{[ik]}$ and symmetric $H_{(ik)}$, have the meaning of *total and proper angular velocity and deformation rate*, respectively; with a more complete meaning than in the ordinary situation. Indeed we have the following expressions:

$$H_{[ik]} = \hat{\Omega}_{ik} + \tilde{\nabla}_{[i}\hat{\gamma}_{k]}, \qquad H_{(ik)} = \hat{K}_{ik} - \hat{\complement}_{(i}\hat{\gamma}_{k)}, \tag{31}$$

which are more general than the usual ones; in particular we can see that *the deformation of the metric tensor $\hat{\gamma}_{ik}$ and the curvature vector of Γ are combined*, according to (30).

It remains, at this point, only to specify the geometric meaning of the spatial rotation coefficients $\hat{\mathcal{R}}_{ik}{}^{j}$; they can be related to the Christoffel symbols relative either to the metric tensor $\hat{\gamma}_{ik}$, or to the metrics γ_{ik} defined by the second equation of (29).

In the first case, we have the following expression:

$$\hat{\mathcal{R}}_{ik}{}^{j} = \gamma^{jh}[\hat{\Gamma}_{ik,h} - (H_{ik} - \tilde{\partial}_i \hat{\gamma}_k)\hat{\gamma}_h + \hat{D}_{hik}], \tag{32}$$

where γ^{jh} is *the reciprocal* of γ_{jh} and $\hat{D}_{hik}$ is defined as follows:

$$\hat{D}_{hik} = \hat{\Omega}_{hi}\hat{\gamma}_k + \hat{\Omega}_{ik}\hat{\gamma}_h - \hat{\Omega}_{kh}\hat{\gamma}_i. \tag{33}$$

It is an expression in which *the difference* $\hat{\mathcal{R}}_{ik}{}^{j} - \hat{\Gamma}_{ik}{}^{j}$ *is not invariant for transformations* (3), because of the presence of the derivatives $\tilde{\partial}_i \hat{\gamma}_k$. A more significant expression is the following:

$$\hat{\mathcal{R}}_{ik}{}^{j} = \Gamma_{ik}{}^{j} + D_{ik}{}^{j}, \tag{34}$$

by means of the *the Christoffel symbols associated to the metric tensor* γ_{ik}:

$$\Gamma_{ik}{}^{j} \stackrel{\text{def}}{=} \tfrac{1}{2}\gamma^{jh}(\tilde{\partial}_i \gamma_{kh} + \tilde{\partial}_k \gamma_{hi} - \tilde{\partial}_h \gamma_{ik}) \tag{35}$$

and of the field:

$$D_{ik}{}^{j} \stackrel{\text{def}}{=} -\gamma^{jh}(H_{(ik)}\hat{\gamma}_h + H_{[kh]}\hat{\gamma}_i - H_{[hi]}\hat{\gamma}_k). \tag{36}$$

We notice that *the last, which is a well-determined function of the metric tensor* γ_{ik}, H_{ik} and $\hat{\gamma}_i$, *has now tensorial character.*

In conclusion, the Ricci rotation coefficients are:

$$\begin{aligned}
&\hat{\mathcal{R}}_{00}{}^{0} = \hat{C}\hat{\gamma}_i, \quad \hat{\mathcal{R}}_{00}{}^{i} = \hat{C}^i, \quad \hat{\mathcal{R}}_{i0}{}^{0} = \hat{H}_i{}^{k}\hat{\gamma}_k, \quad \hat{\mathcal{R}}_{i0}{}^{k} = \hat{\mathcal{R}}_{0i}{}^{k} = \hat{H}_i{}^{k}, \\
&\hat{\mathcal{R}}_{0i}{}^{0} = \hat{\mathsf{C}}_i + \hat{H}_i{}^{k}\hat{\gamma}_k, \quad \hat{\mathcal{R}}_{ik}{}^{0} = H_{ik} - \tilde{\nabla}_i \hat{\gamma}_k, \quad \hat{\mathcal{R}}_{ik}{}^{j} = \Gamma_{ik}{}^{j} + D_{ik}{}^{j},
\end{aligned} \tag{37}$$

where H_{ik} is given by (29) and the Ricci spatial coefficients $\hat{\mathcal{R}}_{ik}{}^{j}$ can be alternatively expressed by (32).

Of course, as for an ordinary frame of reference, where $\hat{\gamma}_i = 0$, also in the general case the transformation law (6) influences all the successive non-holonomic developments. Therefore, *the considered non-holonomic formalism, and the relative geometric-kinematical ingredients, are all invariant for transformations internal to the frame of reference* (3), *i.e. for arbitrary changes of spatial coordinates and of temporal flow.*

5. Kinematical characters of a superfluid

The generalized frame of reference $(\Gamma, \hat{\Sigma})$, previously considered, is equivalent to a couple of ordinary frames: Γ and $\hat{\Gamma}$, respectively, characterized by the flux lines of the two unitary fields $\boldsymbol{\gamma}$ and $\hat{\boldsymbol{\gamma}} = \boldsymbol{\eta}/\eta$. Therefore, such lines constitute *the space–time evolution of a binary fluid mixture*, briefly, a *superfluid*. From this analogy, it follows that the non-holonomic techniques, previously developed, allow to deduce the *geometric-kinematic* *characters of a superfluid*; i.e. the relations between the first order differential characters of the frames Γ and $\hat{\Gamma}$.

Of course, the comparison of the two frames is subordinated to the following *two general properties*:

(a) *every non-holonomic basis adapted to the generalized frame of reference* $(\Gamma, \hat{\Sigma})$ *gives, by duality, a basis adapted to the complementary reference* $(\hat{\gamma}, \Sigma)$; in other words, locally, the basis $\{\tilde{\mathbf{e}}_\alpha\}$ and $\{\tilde{\mathbf{e}}^\alpha\}$ are adapted to the two distinct frames, respectively;

(b) *every tensor of* $\hat{\Sigma}$ *defines an analogous tensor in* Σ *and vice versa*, i.e. the correspondence is biunique; for instance, if we consider the covariant representation, the relative components of the tensor can be interpreted either in $\hat{\Sigma}$ or in Σ, referring to the basis $\{\tilde{\mathbf{e}}_i\}$ or $\{\tilde{\mathbf{e}}^i\}$, respectively.

We notice that the preceding correspondence between the two platforms $\hat{\Sigma}$ and Σ is invariant for transformations (3); however, it *has no isometric character*.

Let us now consider the *gradient of* γ:

$$\nabla\gamma = (H_{ik} + \hat{\gamma}_i C_k)\tilde{\mathbf{e}}^i \otimes \tilde{\mathbf{e}}^k - \gamma \otimes \mathbf{C}, \tag{38}$$

and therefore, the tensor $H_{ik} + \hat{\gamma}_i C_k \in \Sigma$ *summarizes proper deformation and angular velocity* of the frame of reference Γ. According to formulae (30) and (31), we have explicitly

$$k_{ik} = cK_{ik}, \quad \omega_{ik} = c(\hat{\Omega}_{ik} + \tilde{\nabla}_{[i}\hat{\gamma}_{k]} + \hat{\gamma}_{[i}C_{k]}), \quad K_{ik} \overset{\text{def}}{=} \tfrac{1}{2}\partial\gamma_{ik}, \tag{39}$$

where the following *invariance properly* holds:

$$H_{(ik)} = \hat{K}_{ik} - \hat{C}_{(i}\hat{\gamma}_{k)} = K_{ik} - C_{(i}\hat{\gamma}_{k)}. \tag{40}$$

Of course, an expression analogous to (38) holds for the *gradient of* $\hat{\gamma}$:

$$\nabla\hat{\gamma} = \frac{1}{\eta}\hat{\mathcal{R}}_{ik}\hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^k - \hat{\gamma} \otimes \hat{\mathbf{C}}, \tag{41}$$

where the field $\hat{\mathbf{C}}$ is the *curvature vector* of the lines of $\hat{\Gamma}$:

$$\hat{\mathbf{C}} = (C_i + \hat{H}_i{}^k\hat{\gamma}_k - \partial\gamma_i - \hat{\gamma}^k\hat{\mathcal{R}}_{ki})\hat{\mathbf{e}}^i, \tag{42}$$

therefore, the *first order characters of the frame of reference* $\hat{\Gamma}$ are the following:

$$\begin{aligned} \hat{C}_i &= C_i + \hat{\gamma}^k(\eta^2 H_{ik} - H_{ki} + \tilde{\nabla}_k\hat{\gamma}_i) - \partial\hat{\gamma}_i, \\ \hat{k}^{ik} &= (c/\eta)(K_{ik} - C_{(i}\hat{\gamma}_{k)} - \tilde{\nabla}_{(i}\hat{\gamma}_{k)}), \qquad \hat{\omega}_{ik} = (c/\eta)\hat{\Omega}_{ik}. \end{aligned} \tag{43}$$

Such characters are expressed in terms of the fundamental tensors C_i, $\hat{\Omega}_{ik}$ and K_{ik} and the vector $\hat{\gamma}_i$, which characterizes the *difference between the two different platforms* $\hat{\Sigma}$ and Σ; the same ingredients which are present in formula (39), relative to the frame Γ.

Therefore, by elimination, we can deduce the *relations between the characters of the frames* $\hat{\Gamma}$ and Γ [6]:

$$\begin{aligned} \hat{C}_i &= C_i - \partial\hat{\gamma}_i + \hat{\gamma}^k(\eta^2 H_{ik} - H_{ki} + \tilde{\nabla}_k\hat{\gamma}_i), \\ \hat{k}_{ik} + \hat{\omega}_{ik} &= (c/\eta)(H_{ik} - \tilde{\nabla}_i\hat{\gamma}_k), \end{aligned} \tag{44}$$

where the tensor H_{ik}, defined by (31), is expressed, according to (39), by

$$H_{ik} = (1/c)(k_{ik} + \omega_{ik}) - \hat{\gamma}_i C_k. \tag{45}$$

Eqs. (44) give quite general formulae, in the sense that *they hold for every couple of reference frames*, i.e. for *every surperfluid*; moreover this is a *typical relativistic coupling between the different components of the superfluid.*

Of course the *reciprocity principle* also holds, with respect to the exchange of the two congruences Γ and $\hat{\Gamma}$ which, according to (12), involves the exchange of $\hat{\gamma}_i$ with $-\hat{\gamma}_i$.

6. Isometric overlap of $\hat{\Sigma}$ on Σ

To compare the platforms $\hat{\Sigma}$ and Σ, let us briefly examine *induced basis and metrics* on the two platforms from the non-holonomic distribution considered. In $\hat{\Sigma}$ we directly have the *induced basis* $\hat{\mathbf{e}}_i \equiv \tilde{\mathbf{e}}_i$ and the *corresponding metrics* $\hat{\gamma}_{ik}$, as well as their *dual* $\hat{\mathbf{e}}^i$ and $\hat{\gamma}^{ik}$.

As for Σ, we must start from the vectors $\{\tilde{\mathbf{e}}^\alpha\}$, in the sense that $\tilde{\mathbf{e}}^i \equiv \mathbf{e}^i$ represent the *induced basis* in Σ, and so γ^{ik} are the corresponding metrics; subsequently, *by duality in* Σ, we can reproduce the basis $\mathbf{e}_i$ and the metrics γ_{ik}. The following *fundamental relations* hold:

$$\hat{\mathbf{e}}^i = \mathbf{e}^i - \hat{\gamma}^i \boldsymbol{\eta}, \qquad \hat{\mathbf{e}}_i = \mathbf{e}_i - \gamma_i \boldsymbol{\gamma}, \tag{46}$$

and for the two metrics of Σ and $\hat{\Sigma}$ we have, respectively,

$$\gamma_{ik} = \hat{\gamma}_{ik} + \hat{\gamma}_i \hat{\gamma}_k, \qquad \gamma^{ik} = \hat{\gamma}^{ik} - \eta^2 \hat{\gamma}^i \hat{\gamma}^k, \qquad \eta^2 \equiv (1 + \hat{\gamma}^2)^{-1}. \tag{47}$$

Let us now consider the *isometric overlap of* $\hat{\Sigma}$ *on* Σ, i.e. *the correspondence between vectors of the platforms* $\hat{\Sigma}$ *and* $\hat{\Sigma}$ *which preserves the scalar product* (it is a local rotation). It suffices to define, in Σ, the vectors corresponding to the basis $\hat{\mathbf{e}}_i \in \hat{\Sigma}$, which we denote by $\mathbf{d}_i$:

$$\mathbf{d}_i = \Re \hat{\mathbf{e}}_i \equiv \hat{\mathbf{e}}_i + \frac{\eta \hat{\gamma}_i}{1 + \eta}(\boldsymbol{\gamma} + \hat{\boldsymbol{\gamma}}); \tag{48}$$

the following *general formula* holds:

$$\mathbf{d}_i = \left(\delta_i{}^k - \frac{\eta^2}{1 + \eta} \hat{\gamma}_i \hat{\gamma}^k \right) \mathbf{e}_k, \tag{49a}$$

and conversely

$$\mathbf{e}_i = \left(\delta_i{}^k + \frac{\eta}{1 + \eta} \hat{\gamma}_i \hat{\gamma}^k \right) \mathbf{d}_k. \tag{49b}$$

We thus obtain, on the Σ platform, a *second basis* $\{\mathbf{d}_i\}$ coming from $\hat{\Sigma}$ and in one-to-one correspondence with the pre-existing basis $\{\mathbf{e}_i\}$; on the other hand, *the rotation* $\Re$ (with the 2-plane $\hat{\Sigma} \cap \Sigma$ fixed) *preserves the metrics*, i.e. we have the equality

$$\mathbf{d}_i \cdot \mathbf{d}_k = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k = \hat{\gamma}_{ik}, \tag{50}$$

such that Σ *is provided with two different metrics*: γ_{ik} and $\hat{\gamma}_{ik}$, the first natural and the second induced from $\hat{\Sigma}$ by rotation.

Formula (49a) *has a precise kinematical meaning* with respect to the motion of the continuum $\hat{\Gamma}$, relative to the frame Γ. More precisely, by (12) we first deduce the following relation between the vectors $\boldsymbol{\eta}$ and γ:

$$\boldsymbol{\eta} = \gamma - \hat{\gamma}_i \mathbf{e}^i; \tag{51}$$

therefore, denoting by $\hat{\mathbf{V}} = c\hat{\gamma} \equiv c\boldsymbol{\eta}/\eta$ the *local* 4-*velocity* of the continuum $\hat{\Gamma}$, we have the following decomposition:

$$\hat{\mathbf{V}} = \frac{1}{\eta}(c\gamma - c\hat{\gamma}_i \mathbf{e}^i).$$

So we have the *velocity relative to* Γ: $\mathbf{u}$, and the kinematical meaning of the scalar η:

$$\mathbf{u} = -c\hat{\gamma}_i \mathbf{e}^i, \qquad \eta = \sqrt{1 - u^2/c^2}; \tag{52}$$

and (49a) *also takes a kinematical meaning*

$$\mathbf{d}_i = \mathbf{e}_i - \frac{1}{c^2}\frac{u_i}{1+\eta}\mathbf{u}, \qquad u_i \overset{\text{def}}{=} \mathbf{u} \cdot \mathbf{e}_i,$$

by means of the *relative velocity* $\mathbf{u}$. This is a formula which has a perfect analogous in special relativity, where, more generally, we can consider the motion of a continuous system with respect to two different Galileian frames of reference; of course, in our case the continuous system coincides with Γ (*rest frame of reference*) and therefore $\mathbf{v} = 0$.

We notice that formula (49a), which simulates the vectors $\hat{\mathbf{e}}_i$ in Σ, gives us the possibility to examine the two fundamental aspects of the relative kinematics of a scalar particle, i.e. the transformation law of velocity and acceleration in a change of reference frame: *theorems of the relative motion and Coriolis*; we must decompose 4-velocity and 4-acceleration of the particle according to (γ, Σ) and $(\hat{\gamma}, \hat{\Sigma})$, respectively, and then the platform $\hat{\Sigma}$ must be rotated to overlap on Σ, according to (48).

Formula (48) is also fundamental for the *deformation kinematics of relativistic continua*, because the relative *angular and deformation velocities* are combined with the space–time derivatives of the vectors $\mathbf{d}_i$ and $\mathbf{e}_i$.

7. Superfluids and Cosserat continua

What we have seen in Section 6, clearly shows the role of the trihedron $\{\mathbf{d}_i\} \in \Sigma$, in order to specify either the metrics $\hat{\gamma}_{ik}$ or the same platform $\hat{\Sigma}$, which is determined moving backwards, i.e. by antirotating Σ on $\hat{\Sigma}$, according to (51) and to the converse of (48):

$$\hat{\mathbf{e}}_i = \mathbf{d}_i - \hat{\gamma}_i \gamma + \frac{1}{1+\eta}\hat{\gamma}_i \hat{\gamma}_k \mathbf{e}^k \equiv \mathbf{e}_i - \hat{\gamma}_i \gamma. \tag{53}$$

Therefore, *the congruence* $\hat{\Gamma}$ *can be completely characterized, for metrics and position*, in the framework of Γ, *by means of the fundamental trihedron* $\{\mathbf{d}_i\} \in \Sigma$.

Therefore, every superfluid S can be described, in the traditional way, within the frame of reference Γ, although Γ represents only one of the two components of S; we can take into account the *second component (platform* $\hat{\Sigma}$ *and metrics* $\hat{\gamma}_{ik}$*) by associating to* Σ *a determined deformable trihedron* $\{\mathbf{d}_i\}$, depending on the spatial vector $\hat{\gamma}_i$, according to (49a).

From this point of view, a *superfluid becomes a continuous system with internal structure of cosserat type*, in the sense that *every binary mixture can be described by means of only one of the components, for instance* Γ*, provided* Γ *has, at every point, a determined trihedron* $\{\mathbf{d}_i\} \in \Sigma$ *normal to* Γ*, whose motion is independent of the congruence* Γ.

Of course, *the trihedron* $\mathbf{d}_i$ *arises in a natural way* from the non-holonomic distribution (5), and therefore it depends on the coordinates (x^α) chosen, which are adapted to Γ; therefore $\{\mathbf{d}_i\}$ *is not unique*, but defined up to an *arbitrary internal transformation*, since it satisfies the *holonomic transformation law* (second equation of (6)):

$$\mathbf{d}'_i = \frac{\partial x^i}{\partial x^{i'}}\mathbf{d}_i, \tag{54}$$

where *the coefficients are time-independent*, as in the classical situation.

However, while in the classical case *trihedron* $\{\mathbf{d}_i\}$ *can be supposed orthonormal*, since it summarizes a *rigid structure* [8], *in the relativistic framework the deformability of the trihedron is a typical feature* which cannot be removed by means of internal transformations, *neither globally, nor locally* (within some domain).

When compared with a cosserat continuum, a superfluid appears to have easier *dynamical extensions*; of course *the trihedron* $\{\mathbf{d}_i\}$, or its dual on Σ: $\mathbf{d}^i - \hat{\gamma}^{ik}\mathbf{d}_k$, *plays a leading role*, since every tensorial ingredient is decomposed according to γ and $\mathbf{d}_i$: inertial and gravitational forces, stress and couple stress, etc.

In particular, the time derivative $\partial\mathbf{d}_i$, or better the *constrained derivative* (see [1, p. 157]), *gives the angular and deformation velocities of the trihedron*, which are related to the proper ingredients of the continuum Γ.

However, while the analogy between superfluids and cosserat continua, *from the kinematical point of view, has univocal character*, the dynamics of superfluids can be developed from two different points of view. The first way, *assuming the usual evolution equations for both the components*, is immediate:

$$\nabla_\beta T^{\alpha\beta} = 0, \qquad \nabla_\beta \hat{T}^{\alpha\beta} = 0, \tag{55}$$

where the *energy tensors* $T^{\alpha\beta}$ and $\hat{T}^{\alpha\beta}$ *are of fluid type*:

$$\begin{aligned} T^{\alpha\beta} &= \mu_0 V^\alpha V^\beta + p_0\left(g^{\alpha\beta} + \frac{1}{c^2}V^\alpha V^\beta\right), \\ \hat{T}^{\alpha\beta} &= \hat{\mu}_0 \hat{V}^\alpha \hat{V}^\beta + \hat{p}_0\left(g^{\alpha\beta} + \frac{1}{c^2}\hat{V}^\alpha \hat{V}^\beta\right), \end{aligned} \tag{56}$$

and then *translating the equations in terms of cosserat*; i.e. decomposing with respect to γ and the trihedron $\mathbf{d}_i$ and taking into account that $V^i = 0$ for the first fluid, while $\hat{V} \neq 0$ for the second.

If the mixture is not a test system, but generates a gravitational field, we must add to (55) the *Einstein equations*

$$G_{\alpha\beta} = -\chi(T_{\alpha\beta} + \hat{T}_{\alpha\beta}). \tag{57}$$

The second point of view is based on a preliminary hypothesis, namely that *the analogy between superfluids and cosserat continua is also of dynamical type*, in the sense that, for both the systems, the same equations hold.

In this case, the dynamics of a mixture is not any more governed by (55), but by *the general equations of relativistic polar continua* [2], which still constitute *two groups of condition* (for resultants and couples, respectively).

Of course, in this framework, it is not possible to develop the gravitational coupling by means of (57), because cosserat continua are described by two energy tensors, like the gravitational field (*forces* and *couples*); therefore *the Einstein gravitational equations must be suitably modified.*

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